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Tridiagonal pairs of height one

Kazumasa Nomura

College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa 272-0827, Japan

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Abstract

Let (A, A^*) denote a tridiagonal pair on a vector space V over a field $\mathbb{K}$. Let $V_0, \ldots, V_d$ denote a standard ordering of the eigenspaces of A on V, and let $\theta_0, \ldots, \theta_d$ denote the corresponding eigenvalues of A. We assume $d \geqslant 3$. Let q denote a scalar taken from the algebraic closure of $\mathbb{K}$ such that $q^2 + q^{-2} + 1 = (\theta_3 - \theta_0)/(\theta_2 - \theta_1)$. We assume q is not a root of unity. Let ρ_i denote the dimension of V_i. The sequence $\rho_0, \rho_1, \ldots, \rho_d$ is called the *shape* of the tridiagonal pair. It is known there exists a unique integer h $(0 \leqslant h \leqslant d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leqslant i \leqslant h$, $\rho_{i-1} = \rho_i$ for $h < i \leqslant d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \leqslant d$. The integer h is known as the *height* of the tridiagonal pair. In this paper we show that the shape of a tridiagonal pair of height one with $\rho_0 = 1$ is either $1, 2, 2, \ldots, 2, 1$ or $1, 3, 3, 1$. In each case, we display a basis for V and give the action of A, A^* on this basis.

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1. Introduction

The notion of a tridiagonal pair was introduced by Ito et al. [3], generalizing the notion of a Leonard pair which had been introduced by Terwilliger [6]. See

E-mail address: nomura.las@tmd.ac.jp

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Terwilliger's lecture note [8] about Leonard pairs and tridiagonal pairs. A tridiagonal pair is defined as follows.

Definition 1.1 [3]. Let V denote a nonzero finite dimensional vector space over a field $\mathbb{K}$. By a *tridiagonal pair* on V, we mean a pair (A, A^*), where $A : V \longrightarrow V$ and $A^* : V \longrightarrow V$ are linear transformations that satisfy the following conditions.

(i) A and A^* are both diagonalizable on V.
(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d),$$

where $V_{-1} = 0$, $V_{d+1} = 0$.
(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta),$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.
(iv) There is no subspace W of V such that both $AW \subseteq W$, $A^*W \subseteq W$, other than $W = 0$ and $W = V$.

Remark 1.2. With reference to Definition 1.1, it is known that $d = \delta$ [3, Corollary 5.7]. The common number d of the eigenspaces is called the *diameter* of the tridiagonal pair.

Throughout this paper, we fix the following notation. Let $\mathbb{K}$ denote a field and let V denote a nonzero finite dimensional vector space over $\mathbb{K}$. Let (A, A^*) denote a tridiagonal pair on V with diameter $d \geqslant 3$. Let $V_0, V_1, \ldots, V_d$ (respectively $V_0^*, V_1^*, \ldots, V_d^*$) denote an ordering of the eigenspaces of A (respectively A^*) that satisfies the condition (ii) (respectively (iii)) in Definition 1.1. Let ρ_i denote the dimension of V_i. Let θ_i (respectively θ_i^*) denote the eigenvalue of A (respectively A^*) for the eigenspace V_i (respectively V_i^*). Set $\beta = (\theta_3 - \theta_0)/(\theta_2 - \theta_1) - 1$, and let q denote a scalar taken from the algebraic closure $\overline{\mathbb{K}}$ such that $\beta = q^2 + q^{-2}$. We assume q is not a root of unity.

It is known [5, Theorem 3.3] there exists a unique integer h $(0 \leqslant h \leqslant d/2)$ such that $\rho_{i-1} < \rho_i$ for $1 \leqslant i \leqslant h$, $\rho_{i-1} = \rho_i$ for $h < i \leqslant d - h$, and $\rho_{i-1} > \rho_i$ for $d - h < i \leqslant d$. The integer h is known as the *height* of the tridiagonal pair.

Our first main result is the following.

Theorem 1.3. *Suppose $h = 1$ and $\rho_0 = 1$. Then one of the following holds.*

(i) $\rho_0 = 1,\ \rho_1 = \rho_2 = \cdots = \rho_{d-1} = 2,\ \rho_d = 1,$
(ii) $d = 3,\ \rho_0 = 1,\ \rho_1 = \rho_2 = 3,\ \rho_3 = 1.$

In each case of (i), (ii), we display a basis for V and give the action of A, A^* on this basis. In order to do this we review some more definitions. Set

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d)$$

for $0 \leqslant i \leqslant d$. By [3] we have

$$V = U_0 + U_1 + \cdots + U_d \quad \text{(direct sum)},$$

and

$$(A - \theta_i I)U_i \subseteq U_{i+1}, \quad (A^* - \theta_i^* I)U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d),$$

where we set $U_{-1} = U_{d+1} = 0$. The sequence $U_0, U_1, \ldots, U_d$ is called the *split decomposition* of (A, A^*). The *raising map* R and the *lowering map* L are defined by

$$R = A - \sum_{i=0}^{d} \theta_i F_i, \qquad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i,$$

where $F_i : V \longrightarrow U_i$ denotes the projection. The maps R, L satisfy

$$RU_i \subseteq U_{i+1}, \qquad LU_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d).$$

It is known [3] that the eigenvalues are represented as

$$\theta_i = aq^{2i} + bq^{-2i} + c \quad (0 \leqslant i \leqslant d), \tag{1}$$
$$\theta_i^* = a^*q^{2i} + b^*q^{-2i} + c^* \quad (0 \leqslant i \leqslant d), \tag{2}$$

for some scalars a, a^*, b, b^*, c, c^* in $\overline{\mathbb{K}}$. We set

$$\eta_i = (q - q^{-1})^3(aa^*q^i - bb^*q^{-i}).$$

We use the following notation;

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad \begin{bmatrix} n \\ 2 \end{bmatrix} = \frac{[n][n-1]}{[2]}.$$

We now define our basis for the case (i) in Theorem 1.3.

Definition 1.4. With reference to Theorem 1.3 (i), let u denote a nonzero vector in U_0, and define $u_i = R^i u$ for $0 \leqslant i \leqslant d$. As we will show in Section 4, there exists a unique vector $v \in U_1$ such that (i) $R^{d-1}v = 0$; and (ii) $v - Lu_2$ is a scalar multiple of u_1. We define $v_i = R^{i-1}v$ for $1 \leqslant i \leqslant d-1$.

Theorem 1.5. *Suppose Theorem* 1.3(i) *holds. Then*

(i) u_0 *is a basis for* U_0,
(ii) u_i, v_i *is a basis for* U_i $(1 \leqslant i \leqslant d-1)$,

(iii) u_d *is a basis for* U_d,
(iv) *the vectors*

$$u_0,\ u_1,\ v_1,\ \ldots,\ u_{d-1},\ v_{d-1},\ u_d \tag{3}$$

form a basis for V.

We now give the action of R, L on the basis in Theorem 1.5.

Theorem 1.6. *Suppose Theorem* 1.3(i) *holds. Then there exist scalars* $\lambda,\ \mu$ *in* $\overline{\mathbb{K}}$ *such that the maps* R, L *act on the basis* (3) *as follows.*

$$\begin{aligned}
&Ru_i = u_{i+1} \quad (0 \leqslant i \leqslant d-1), \quad Ru_d = 0,\\
&Rv_i = v_{i+1} \quad (1 \leqslant i \leqslant d-2), \quad Rv_{d-1} = 0,\\
&Lu_0 = 0, \quad Lu_1 = a_0u_0, \quad Lu_{i+1} = a_iu_i + b_iv_i \quad (1 \leqslant i \leqslant d-1),\\
&Lv_1 = e_0u_0, \quad Lv_{i+1} = e_iu_i + c_iv_i \quad (1 \leqslant i \leqslant d-2),
\end{aligned}$$

where

$$\begin{aligned}
&a_i = [i+1][d-i](\lambda - [i]\eta_{d+i+1}) \quad (0 \leqslant i \leqslant d-1),\\
&b_i = \begin{bmatrix} i+1 \\ 2 \end{bmatrix} \quad (1 \leqslant i \leqslant d-1),\\
&c_i = [i][d-i-1](\mu - [i-1]\eta_{d+i+2}) \quad (1 \leqslant i \leqslant d-2),\\
&e_i = \begin{bmatrix} d-i \\ 2 \end{bmatrix}\left(-\lambda^2 - \mu^2 + \frac{[4]}{[2]}\lambda\mu + [2]\eta_{d+3}\lambda - [2]\eta_{d+1}\mu\right) \quad (0 \leqslant i \leqslant d-2).
\end{aligned}$$

Remark 1.7. The parameters λ, μ are not necessarily in $\mathbb{K}$. However since $[d]\lambda = a_0$ and $[d-2]\mu = c_1$, we find $[d]\lambda$ and $[d-2]\mu$ are in $\mathbb{K}$.

Theorem 1.8. *Suppose Theorem* 1.3(i) *holds. Then the maps* A, A^* *act on the basis* (3) *as follows.*

$$\begin{aligned}
&Au_i = \theta_iu_i + u_{i+1} \quad (0 \leqslant i \leqslant d-1), \quad Au_d = \theta_du_d,\\
&Av_i = \theta_iv_i + v_{i+1} \quad (1 \leqslant i \leqslant d-2), \quad Av_{d-1} = \theta_{d-1}v_{d-1},\\
&A^*u_0 = \theta^*_0u_0, \quad A^*u_1 = a_0u_0 + \theta^*_1u_1,\\
&A^*u_{i+1} = a_iu_i + b_iv_i + \theta^*_{i+1}u_{i+1} \quad (1 \leqslant i \leqslant d-1),\\
&A^*v_1 = e_0u_0 + \theta^*_1v_1, \quad A^*v_{i+1} = e_iu_i + c_iv_i + \theta^*_{i+1}v_{i+1} \quad (1 \leqslant i \leqslant d-2).
\end{aligned}$$

Theorem 1.9. *Let* a_0, c_1, a, a^*, b, b^*, c, c^*, q *denote scalars in* $\mathbb{K}$. *Let* V *denote a vector space over* $\mathbb{K}$ *with dimension* $2d$ $(d \geqslant 3)$, *and let* $A,\ A^* : V \longrightarrow V$ *denote linear transformations which act on some basis* $u_0, u_1, v_1, \ldots, u_{d-1}, v_{d-1}, u_d$ *as in Theorems* 1.6 *and* 1.8. *Further suppose that* V *is irreducible as an* (A, A^*)-*module. Then* $(A,\ A^*)$ *is a tridiagonal pair on* V.

Next we consider the case (ii) in Theorem 1.3.

Definition 1.10. With reference to Theorem 1.3(ii), we define u_0, u_1, u_2, u_3 and v_1, v_2 as in Definition 1.4. As we will show in Section 4, there exists a unique vector $w \in U_1$ such that (i) $R^2 w = 0$; and (ii) $w - Lv_2$ is a scalar multiple of u_1. We define $w_1 = w$, $w_2 = Rw$.

Theorem 1.11. *Suppose Theorem* 1.3(i) *holds. Then*

(i) u_0 *is a basis for* U_0,
(ii) u_i, v_i, w_i *is a basis for* U_i $(1 \leqslant i \leqslant 2)$,
(iii) u_3 *is a basis for* U_3,
(iv) *the vectors*

$$u_0,\ u_1,\ v_1,\ w_1,\ u_2,\ v_2,\ w_2,\ u_3 \tag{4}$$

form a basis for V.

We now give the action of R, L on the basis in Theorem 1.11.

Theorem 1.12. *Suppose Theorem* 1.3(ii) *holds. Then there exist scalars* a_0, e_1, f_1 *in* $\mathbb{K}$ *such that the maps* R, L *act on the basis* (4) *as follows.*

$$\begin{aligned}
&Ru_0 = u_1, \quad Ru_1 = u_2, \quad Ru_2 = u_3, \quad Ru_3 = 0,\\
&Rv_1 = v_2, \quad Rv_2 = 0,\\
&Rw_1 = w_2, \quad Rw_2 = 0,\\
&Lu_0 = 0, \quad Lu_1 = a_0 u_0, \qquad Lu_{i+1} = a_i u_i + b_i v_i \quad (1 \leqslant i \leqslant 2),\\
&Lv_1 = e_0 u_0, \quad Lv_2 = e_1 u_1 + w_1,\\
&Lw_1 = f_0 u_0, \quad Lw_2 = f_1 u_1 + s_1 v_1 + t_1 w_1,
\end{aligned}$$

where

$$\begin{aligned}
&a_1 = [2][2](\lambda - \eta_5), \quad a_2 = [3](\lambda - [2]\eta_6),\\
&b_1 = 1, \quad b_2 = [3],\\
&e_0 = [3]e_1, \quad f_0 = [3]f_1,\\
&s_1 = -\lambda^2 - e_1 + [2]\eta_6\lambda,\\
&t_1 = \frac{[4]\lambda}{[2]} - [2]\eta_4,
\end{aligned}$$

and where

$$\lambda = \frac{a_0}{[3]}.$$

Theorem 1.13. *Suppose Theorem* 1.3(ii) *holds. Then the maps* A, A^* *act on the basis* (4) *as follows.*

$$
\begin{aligned}
&Au_i = \theta_i u_i + u_{i+1} \quad (0 \leqslant i \leqslant 2), \quad Au_3 = \theta_3 u_3,\\
&Av_1 = \theta_1 v_1 + v_2, \quad Av_2 = \theta_2 v_2,\\
&A^* u_0 = \theta_0^* u_0, \quad A^* u_1 = a_0 u_0 + \theta_1^* u_1,\\
&A^* u_{i+1} = a_i u_i + b_i v_i + \theta_{i+1}^* u_{i+1} \quad (1 \leqslant i \leqslant 2),\\
&A^* v_1 = e_0 u_0 + \theta_1^* v_1, \quad A^* v_2 = e_1 u_1 + w_1 + \theta_2^* v_2,\\
&A^* w_1 = f_0 u_0 + \theta_1^* w_1, \quad A^* w_2 = f_1 u_1 + s_1 v_1 + t_1 w_1 + \theta_2^* w_2.
\end{aligned}
$$

Theorem 1.14. *Let $a_0, e_1, f_1, a, a^*, b, b^*, c, c^*, q$ denote scalars in $\mathbb{K}$. Let V denote a vector space over $\mathbb{K}$ with dimension 8, and let $A, A^* : V \longrightarrow V$ denote linear transformations which act on some basis $u_0, u_1, v_1, w_1, u_2, v_2, w_2, u_3$ as in Theorems 1.12 and 1.13. Further suppose that V is irreducible as an (A, A^*)-module. Then (A, A^*) is a tridiagonal pair on V.*

Remark 1.15. There are some works by Alnajjar and Curtin for some family of tridiagonal pairs which satisfy Theorem 1.3(i). In [1], they give the action of A, A^* on some basis, which is different from the basis (3). They assume the tridiagonal pair is of q-Serre type, i.e., $aa^* = 0$, $bb^* = 0$, $c = c^* = 0$ hold in (1), (2). Further they assume some technical condition. So our results are more general. In [2], they give an action of the affine quantum group $U_q(\widehat{sl}(2))$ on V.

Remark 1.16. It is known [4] that

$$
\rho_i \leqslant \binom{i}{2} \quad (0 \leqslant i \leqslant d) \tag{5}
$$

holds for tridiagonal pairs of q-Serre type with $\mathbb{K}$ algebraically closed. It is conjectured [4] that (5) holds for all tridiagonal pairs when $\mathbb{K}$ is algebraically closed.

Remark 1.17. With reference to Theorem 1.6, let φ_i denote the eigenvalue of $L^i R^i$ on U_0 $(0 \leqslant i \leqslant d)$. Then the parameters λ, μ can be written in terms of $\varphi_1, \varphi_2, \varphi_3$ as follows:

$$
\begin{aligned}
\lambda &= \frac{\varphi_1}{[d]},\\
\mu &= \frac{\varphi_3 - [3]\,[d-2]\left(-\frac{[2][d-1]}{[d][d]}\,\varphi_1^3 + \frac{[2][d-1]}{[d]}\,\eta_{d+2}\,\varphi_1^2 + \frac{2}{[d]}\,\varphi_1\,\varphi_2 - [2]\,\eta_{d+3}\,\varphi_2\right)}{[3][d-2]\left(-\frac{[2][d-1]}{[d]}\,\varphi_1^2 + [2]\,[d-1]\,\eta_{d+2}\,\varphi_1 + \varphi_2\right)}.
\end{aligned}
$$

We remark that the denominator is equal to $[3][d-2]e_0$ and it is nonzero.

In Section 2, we pick up some basic results from [3]. In the proof of Theorem 1.3, we use the refined split decomposition, which was introduced in [5]. In Section 3, we recall about the refined split decomposition. In Section 4, we construct a basis and determine the action of L. In Sections 5 and 6, we prove Theorem 1.3. The proofs of Theorems 1.5–1.9 are given in Section 7. Theorems 1.11–1.14 can be shown in a similar way, so we omit the proofs.

2. Background

In this section, we recall some known facts about the tridiagonal pairs.

For $0 \leqslant i \leqslant d$, we set

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d). \tag{6}$$

Lemma 2.1 [3, Theorem 4.6]. *The space V is decomposed as*

$$V = U_0 + U_1 + \cdots + U_d \quad (\textit{direct sum}). \tag{7}$$

The decomposition given in (7) is called the *split decomposition* of the tridiagonal pair.

Lemma 2.2 [3, Corollary 5.7]. *For $0 \leqslant i \leqslant d$,*

(i) $\dim V_i = \dim V_i^* = \dim U_i = \rho_i$,
(ii) $\rho_i = \rho_{d-i}$.

Let $F_i : V \to U_i$ denote the projection with respect to the direct sum (7). Then for $0 \leqslant i, j \leqslant d$,

$$F_0 + F_1 + \cdots + F_d = I, \quad F_i F_i = F_i, \quad F_i F_j = 0 \quad \text{if } i \neq j. \tag{8}$$

The *raising map* R and the *lowering map* L are defined as follows.

$$R = A - \sum_{i=0}^{d} \theta_i F_i, \qquad L = A^* - \sum_{i=0}^{d} \theta_i^* F_i. \tag{9}$$

Lemma 2.3 [3, Corollary 6.3]

(i) $RU_i \subseteq U_{i+1} \quad (0 \leqslant i \leqslant d-1), \quad RU_d = 0.$
(ii) $LU_i \subseteq U_{i-1} \quad (1 \leqslant i \leqslant d), \quad LU_0 = 0.$

Lemma 2.4. *Let W denote a subspace of V. Suppose that $RW \subseteq W$, $LW \subseteq W$ and $F_i W \subseteq W$ for $0 \leqslant i \leqslant d$. Then $W = 0$ or $W = V$.*

Proof. Observe that A and A^* are represented as linear combinations of R, L, F_i $(0 \leqslant i \leqslant d)$ by (9), so that $AW \subseteq W$ and $A^*W \subseteq W$. Now the result follows from Definition 1.1(iv). $\square$

Lemma 2.5 [3, Theorem 10.1]. *There is a sequence of scalars* β, γ, γ^*, ϱ, ϱ^* *taken from* $\mathbb{K}$ *such that*

$$[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0, \tag{10}$$

$$[A^*,\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^* A] = 0, \tag{11}$$

where $[B, C] = BC - CB$. *The sequence is unique if the diameter is at least three.*

The above relations are known as the *tridiagonal relations*. These relations imply the following relations between R and L. Let ε_i $(0 \leqslant i \leqslant d-2)$ denote the scalar defined by

$$\varepsilon_i = (\theta_i - \theta_{i+2})(\theta^*_{i+1} - \theta^*_{i+2}) - (\theta^*_{i+2} - \theta^*_i)(\theta_{i+1} - \theta_i). \tag{12}$$

Lemma 2.6 [3, Theorem 12.1]. *For* $0 \leqslant i \leqslant d-2$,

$$(R^3L - (\beta+1)R^2LR + (\beta+1)RLR^2 - LR^3 + (\beta+1)\varepsilon_i R^2)F_i = 0, \tag{13}$$

$$(L^3R - (\beta+1)L^2RL + (\beta+1)LRL^2 - RL^3 - (\beta+1)\varepsilon_i L^2)F_{i+2} = 0. \tag{14}$$

Lemma 2.7. *Let* V *denote a vector space over* $\mathbb{K}$. *Suppose* V *is decomposed into direct sum of subspaces* $U_0, U_1, \ldots, U_d$ $(d \geqslant 3)$, *and let* F_i *denote the projection onto* F_i. *Let* β *denote a scalar in* $\mathbb{K}$, *and let* $\theta_0, \theta_1, \ldots, \theta_d$ (*respectively* $\theta^*_0, \ldots, \theta^*_d$) *denote distinct scalars in* $\mathbb{K}$ *such that the expressions*

$$\frac{\theta_{i+3} - \theta_i}{\theta_{i+2} - \theta_{i+1}}, \qquad \frac{\theta^*_{i+3} - \theta^*_i}{\theta^*_{i+2} - \theta^*_{i+1}}$$

both equal to $\beta + 1$ *for* $0 \leqslant i \leqslant d-3$. *Define scalars* γ, ϱ, ε_i $(0 \leqslant i \leqslant d-2)$ *by*

$$\begin{aligned}
\gamma &= \theta_0 - \beta\theta_1 + \theta_2,\\
\varrho &= \theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1),\\
\varepsilon_i &= (\theta_i - \theta_{i+2})(\theta^*_{i+1} - \theta^*_{i+2}) - (\theta^*_{i+2} - \theta^*_i)(\theta_{i+1} - \theta_i).
\end{aligned}$$

Let $R, L : V \longrightarrow V$ *denote linear transformations such that* $RU_i \subseteq U_{i+1}$ *and* $LU_i \subseteq U_{i-1}$ *holds for* $0 \leqslant i \leqslant d$, *where we set* $U_{-1} = U_{d+1} = 0$. *Suppose* R, L *satisfy* (13) *for* $0 \leqslant i \leqslant d-2$. *Define maps* A, A^* *by*

$$A = R + \sum_{i=0}^{d} \theta_i F_i, \qquad A^* = L + \sum_{i=0}^{d} \theta^*_i F_i.$$

Then A, A^* *satisfy* (10).

Proof. Let C denote the left side of (10). Replace A (respectively A^*) in each term of C by $R + \sum_{i=0}^{d} \theta_i F_i$ (respectively $L + \sum_{i=0}^{d} \theta_i^* F_i$). After expanding each term of CF_j $(0 \leqslant j \leqslant d-2)$, collect the resulting expression in R, L, and verify that each term vanishes. $\square$

Lemma 2.8 [7, Theorem 3.10]. *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote scalars in $\mathbb{K}$, and assume q is not a root of unity, where $\beta = q^2 + q^{-2}$. Let T denote a the algebra generated by two symbols A, A^* subject to the relations* (10), (11). *Let V denote an irreducible finite dimensional T-module and assume each of A, A^* is diagonalizable on V. Then A, A^* act on V as a tridiagonal pair.*

3. The refined split decomposition

In this section, we pick up some results concerning the refined split decomposition from [5]. For the rest of this paper, let h denote the height of the tridiagonal pair.

For $0 \leqslant r \leqslant h$ and $r \leqslant i \leqslant d-r$, we set

$$U_i^{(r)} = R^{i-r}(U_r \cap \text{ Ker } R^{d-2r+1}). \tag{15}$$

Lemma 3.1 [5, Lemma 4.1]. *The following hold for $0 \leqslant r \leqslant h$.*

(i) $U_0^{(0)} = U_0$ *and* $U_d^{(0)} = U_d$.
(ii) $U_i^{(r)} \subseteq U_i$ $(r \leqslant i \leqslant d-r)$.
(iii) $U_r^{(r)} = U_r \cap \operatorname{Ker} R^{d-2r+1}$.
(iv) $U_i^{(r)} = R^{i-r} U_r^{(r)}$ $(r \leqslant i \leqslant d-r)$.
(v) $RU_i^{(r)} = U_{i+1}^{(r)}$ $(r \leqslant i \leqslant d-r-1)$, $RU_{d-r}^{(r)} = 0$.
(vi) *The restriction* $R|_{U_i^{(r)}} : U_i^{(r)} \longrightarrow U_{i+1}^{(r)}$ *is a bijection* $(r \leqslant i \leqslant d-r-1)$.

Lemma 3.2 [5, Lemma 4.3]. *For* $0 \leqslant r \leqslant h$,

$$\dim U_i^{(r)} = \rho_r - \rho_{r-1} \quad (r \leqslant i \leqslant d-r), \tag{16}$$

where we set $\rho_{-1} = 0$.

Lemma 3.3 [5, Lemma 4.7]. *For* $0 \leqslant i \leqslant d$,

$$U_i = \sum_{r=0}^{m} U_i^{(r)} \quad (\textit{direct sum}), \tag{17}$$

where $m = \min\{i, h, d-i\}$.

For $0 \leqslant r \leqslant h$, we set

$$U^{(r)} = \sum_{i=r}^{d-r} U_i^{(r)}. \tag{18}$$

Lemma 3.4 [5, Lemma 5.1]. *V is decomposed as*

$$V = \sum_{r=0}^{h} U^{(r)} \quad (\textit{direct sum}). \tag{19}$$

Lemma 3.5 [5, Lemma 5.2]. *For $0 \leqslant r \leqslant h$ and $0 \leqslant i \leqslant d$,*

$$U^{(r)} \cap U_i = \begin{cases} U_i^{(r)} & \textit{if } r \leqslant i \leqslant d-r, \\ 0 & \textit{otherwise}. \end{cases} \tag{20}$$

Lemma 3.6 [5, Lemma 5.3]. *For $0 \leqslant r \leqslant h$,*

$$RU^{(r)} \subseteq U^{(r)}. \tag{21}$$

Lemma 3.7 [5, Theorem 5.6]. *For $0 \leqslant r \leqslant h$,*

$$LU^{(r)} \subseteq U^{(r-1)} + U^{(r)} + U^{(r+1)}, \tag{22}$$

where we set $U^{(-1)} = U^{(h+1)} = 0$.

Let

$$F^{(r)} : V \longrightarrow U^{(r)} \quad (0 \leqslant r \leqslant h)$$

denote the projection with respect to the direct sum $V = \sum_{r=0}^{h} U^{(r)}$. Observe that for $0 \leqslant r \leqslant h$ and $0 \leqslant s \leqslant h$,

$$\begin{aligned} &F^{(0)} + F^{(1)} + \cdots + F^{(h)} = I, \qquad F^{(r)}F^{(r)} = F^{(r)}, \\ &F^{(r)}F^{(s)} = 0 \quad \text{if } r \neq s. \end{aligned} \tag{23}$$

We set

$$F_i^{(r)} = F_i F^{(r)} \quad (0 \leqslant r \leqslant h, \ 0 \leqslant i \leqslant d).$$

Lemma 3.8 [5, Lemma 6.1]. *For $0 \leqslant r \leqslant h$ and $0 \leqslant i \leqslant d$,*

(i) $F_i^{(r)} = F^{(r)}F_i = F_i F^{(r)}$,
(ii) $F_0^{(0)} = F_0$ *and* $F_d^{(0)} = F_d$,
(iii) $F_i^{(r)} \neq 0$ *if and only if* $r \leqslant i \leqslant d-r$.

Lemma 3.9 [5, Lemma 6.2]. *For* $0 \leqslant r \leqslant h$ *and* $r \leqslant i \leqslant d-r$, $F_i^{(r)}V = U_i^{(r)}$, *and*

$$F_i^{(r)} : V \longrightarrow U_i^{(r)}$$

is the projection with respect to the direct sum $V = \sum_{r=0}^{h} \sum_{i=r}^{d-r} U_i^{(r)}$.

Lemma 3.10 [5, Lemma 6.3]. *For* $0 \leqslant r \leqslant h$.

(i) $F^{(r)}R = RF^{(r)}$,
(ii) $F_i^{(r)}R = RF_{i-1}^{(r)}$ $(1 \leqslant i \leqslant d)$,
(iii) $RF_r^{(r)} = 0$.

We set

$$L^{(-)} = \sum_{r=1}^{h} F^{(r-1)}LF^{(r)}, \quad L^{(0)} = \sum_{r=0}^{h} F^{(r)}LF^{(r)},$$

$$L^{(+)} = \sum_{r=0}^{h-1} F^{(r+1)}LF^{(r)}. \tag{24}$$

Lemma 3.11 [5, Lemma 6.5]

$$L = L^{(-)} + L^{(0)} + L^{(+)}. \tag{25}$$

Lemma 3.12 [5, Lemma 6.6]. *The following hold.*

(i) $F^{(r-1)}LF^{(r)} = L^{(-)}F^{(r)}$ $(1 \leqslant r \leqslant h)$,
(ii) $F^{(r)}LF^{(r)} = L^{(0)}F^{(r)}$ $(0 \leqslant r \leqslant h)$,
(iii) $F^{(r+1)}LF^{(r)} = L^{(+)}F^{(r)}$ $(0 \leqslant r \leqslant h-1)$.

Lemma 3.13 [5, Lemma 6.7]. *The following hold for* $0 \leqslant r \leqslant h$.

(i) $L^{(0)}F_r^{(r)} = 0$.
(ii) $L^{(+)}F_r^{(r)} = L^{(+)}F_{r+1}^{(r)} = 0$.

Lemma 3.14 [5, Theorem 8.4]. *For* $0 \leqslant r \leqslant h-1$ *and* $r+2 \leqslant i \leqslant d-r-1$,

$$RL^{(+)} - \frac{[i-r-1]}{[i-r+1]}L^{(+)}R$$

vanishes on $U_i^{(r)}$.

Lemma 3.15 [5, Theorem 9.4]. *For* $1 \leqslant r \leqslant h$ *and* $r \leqslant i \leqslant d-r-1$,

$$RL^{(-)} - \frac{[d-r-i+2]}{[d-r-i]} L^{(-)} R$$

vanishes on $U_i^{(r)}$.

Lemma 3.16 [3, Lemma 8.6, Theorem 11.1]. *There are scalars* a, a^*, b, b^*, c, c^* *in the algebraic closure* $\overline{\mathbb{K}}$ *such that*

$$\theta_i = aq^{2i} + bq^{-2i} + c \quad (0 \leqslant i \leqslant d), \tag{26}$$

$$\theta_i^* = a^*q^{2i} + b^*q^{-2i} + c^* \quad (0 \leqslant i \leqslant d). \tag{27}$$

We fix scalars a, a^*, b, b^*, c, c^* which satisfy (26) and (27).

Lemma 3.17 [5, Lemma 10.7]. *For* $0 \leqslant i \leqslant d-2$,

$$\varepsilon_i = [2](q-q^{-1})^3(aa^*q^{4i+4} - bb^*q^{-(4i+4)}). \tag{28}$$

We set

$$\mu_i = (q-q^{-1})^3(aa^*q^{d+2i} - bb^*q^{-d-2i}).$$

Lemma 3.18 [5, Theorem 10.9]. *For* $0 \leqslant r \leqslant h$ *and* $r+1 \leqslant i \leqslant d-r-1$, *the following map vanishes on* $U_i^{(r)}$;

$$RL^{(0)} - \frac{[i-r][d-r-i+1]}{[i-r+1][d-r-i]} L^{(0)} R - [i-r][d-r-i+1]\mu_i I. \tag{29}$$

Lemma 3.19. *Let* r *denote an integer with* $0 \leqslant r \leqslant h$. *Let* Y *denote a subspace of* $U_r^{(r)}$ *such that* $L^{(0)}RY \subseteq Y$. *We set* $W = \sum_{i=0}^{d-2r} R^i Y$. *Then* $L^{(0)}W \subseteq W$.

Proof. We show

$$L^{(0)} R^i Y \subseteq R^{i-1} Y \quad (1 \leqslant i \leqslant d-2r), \tag{30}$$

by induction. Clearly (30) holds for $i = 1$ by our assumption. Assume $2 \leqslant i \leqslant d - 2r$. Pick any vector u in Y and observe that $R^{i-1}u$ belongs to $U_{r+i-1}^{(r)}$ by Lemma 3.1. Applying Lemma 3.18 to $R^{i-1}u$,

$$\begin{aligned} L^{(0)} R^i u = & \frac{[i][d-2r-i+1]}{[i-1][d-2r-i+2]} RL^{(0)} R^{i-1} u \\ & - [i][d-2r-i+1]\mu_{r+i-1} R^{i-1} u. \end{aligned}$$

This implies that $L^{(0)}R^i u$ lies in the span of $\{RL^{(0)}R^{i-1}u,\ R^{i-1}u\}$, where we have $R^{i-1}u \in R^{i-1}Y$, and $L^{(0)}R^{i-1}u \in R^{i-2}Y$ by induction. Hence $L^{(0)}R^i u$ belongs to $R^{i-1}Y$. We have shown (30). We have also $L^{(0)}Y \subseteq L^{(0)}U_r^{(r)} = 0$ by Lemma 3.13. Thus $L^{(0)}W \subseteq W$. $\square$

4. Determining the action of *L*

For the rest of this paper, we assume $\rho_0 = \rho_d = 1$ and $\rho_1 = \rho_2 = \cdots = \rho_{d-1} \geqslant 2$, so that $h = 1$.

Lemma 4.1. *V is decomposed as $V = \sum_{i=0}^{d} U_i^{(0)} + \sum_{i=1}^{d-1} U_i^{(1)}$ (direct sum).*

Proof. Follows from (18) and (19) with $h = 1$. $\square$

We fix a nonzero vector u_0 in U_0, and we set $u_i = R^i u_0$ $(1 \leqslant i \leqslant d)$.

Lemma 4.2. *For $0 \leqslant i \leqslant d$, $\{u_i\}$ is a basis of $U_i^{(0)}$.*

Proof. Follows from Lemma 3.1. $\square$

Thus $L^{(0)}u_{i+1}$ is a scalar multiple of u_i. We set

$$L^{(0)}u_{i+1} = a_i u_i \quad (0 \leqslant i \leqslant d-1). \tag{31}$$

Lemma 4.3. *For $0 \leqslant i \leqslant d-1$,*

$$a_i = [i+1][d-i]\left(\frac{a_0}{[d]} - \sum_{k=1}^{i}\mu_k\right). \tag{32}$$

Proof. We show (32) by induction. Clearly (32) holds for $i = 0$, so we assume $1 \leqslant i \leqslant d-1$. Applying Lemma 3.18 to u_i,

$$L^{(0)}Ru_i = \frac{[i+1][d-i]}{[i][d-i+1]}RL^{(0)}u_i - [i+1][d-i]\mu_i u_i,$$

where we have $L^{(0)}Ru_i = L^{(0)}u_{i+1} = a_i u_i$ and $RL^{(0)}u_i = R(a_{i-1}u_{i-1}) = a_{i-1}u_i$. Hence

$$a_i = \frac{[i+1][d-i]}{[i][d-i+1]}a_{i-1} - [i+1][d-i]\mu_i.$$

By induction,

$$a_{i-1} = [i][d-i+1]\left(\frac{a_0}{[d]} - \sum_{k=1}^{i-1}\mu_k\right).$$

Now (32) follows. □

Lemma 4.4

(i) $L^{(+)}u_0 = 0$, $L^{(+)}u_1 = 0$,
(ii) $L^{(+)}u_2 \neq 0$.

Proof. (i) Follows from Lemma 3.13.
(ii) Suppose $L^{(+)}u_2 = 0$. Applying Lemma 3.14 to u_i,

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = \frac{[i+1]}{[i-1]}RL^{(+)}u_i \quad (2 \leqslant i \leqslant d-1).$$

Combining with (i), this implies $L^{(+)}u_i = 0$ for $0 \leqslant i \leqslant d$, so that $L^{(+)}U^{(0)} = 0$. Using (24) and (25), this implies

$$LU^{(0)} = L^{(-)}U^{(0)} + L^{(0)}U^{(0)} + L^{(+)}U^{(0)} = L^{(0)}U^{(0)} \subseteq U^{(0)}.$$

Thus $U^{(0)}$ is invariant under L. Also we have $RU^{(0)} \subseteq U^{(0)}$ by Lemma 3.6, and $F_iU^{(0)} = U_i^{(0)} \subseteq U^{(0)}$ $(0 \leqslant i \leqslant d)$ by Lemma 3.8. These imply $U^{(0)} = V$ by Lemma 2.4, a contradiction. □

We set $v_1 = L^{(+)}u_2$, $v_i = R^{i-1}v_1$ $(2 \leqslant i \leqslant d-1)$.

Lemma 4.5. *For* $1 \leqslant i \leqslant d-1$, v_i *lies in* $U_i^{(1)}$, *and* $v_i \neq 0$.

Proof. Follows from Lemmas 4.4 and 3.1. □

We set

$$b_i = \begin{bmatrix} i+1 \\ 2 \end{bmatrix} \quad (1 \leqslant i \leqslant d-1). \tag{33}$$

Lemma 4.6. *For* $1 \leqslant i \leqslant d-1$,

$$L^{(+)}u_{i+1} = b_iv_i. \tag{34}$$

Proof. We show (34) by induction. Clearly (34) holds for $i = 1$, so we assume $2 \leqslant i \leqslant d-1$. Applying Lemma 3.14 to u_i,

$$L^{(+)}u_{i+1} = L^{(+)}Ru_i = \frac{[i+1]}{[i-1]}RL^{(+)}u_i.$$

By induction,

$$L^{(+)}u_i = b_{i-1}v_{i-1} = \begin{bmatrix} i \\ 2 \end{bmatrix} v_{i-1}.$$

Hence

$$L^{(+)}u_{i+1} = \frac{[i+1]}{[i-1]}\begin{bmatrix} i \\ 2 \end{bmatrix} Rv_{i-1} = \begin{bmatrix} i+1 \\ 2 \end{bmatrix} v_i. \qquad \square$$

Observe that $L^{(-)}v_{i+1}$ is a scalar multiple of u_i by Lemmas 4.5 and 4.2, so we may write

$$L^{(-)}v_{i+1} = e_i u_i \quad (0 \leqslant i \leqslant d-2). \tag{35}$$

Lemma 4.7. *For* $0 \leqslant i \leqslant d-2$,

$$e_i = \frac{[d-i][d-i-1]}{[d][d-1]}e_0. \tag{36}$$

Proof. We show (36) by induction. Clearly (36) holds for $i = 0$, so we assume $1 \leqslant i \leqslant d-2$. Applying Lemma 3.15 to v_i,

$$L^{(-)}Rv_i = \frac{[d-i-1]}{[d-i+1]}RL^{(-)}v_i,$$

where we have

$$L^{(-)}Rv_i = L^{(-)}v_{i+1} = e_i u_i,$$

and

$$RL^{(-)}v_i = R(e_{i-1}u_{i-1}) = e_{i-1}u_i,$$

so that

$$e_i = \frac{[d-i-1]}{[d-i+1]}e_{i-1}.$$

By induction

$$e_{i-1} = \frac{[d-i+1][d-i]}{[d][d-1]}e_0.$$

Now (36) follows. $\square$

Lemma 4.8. *Suppose* $\rho_1 = 2$. *Then* $\{v_i\}$ *is a basis of* $U_i^{(1)}$ $(1 \leqslant i \leqslant d-1)$.

Proof. Follows from Lemmas 4.8 and 3.1. $\square$

Hence, when $\rho_1 = 2$,

$$L^{(0)}v_1 = 0, \qquad L^{(0)}v_{i+1} = c_i v_i \quad (1 \leqslant i \leqslant d-2) \tag{37}$$

hold for some scalars $c_1, \ldots, c_{d-2}$.

Lemma 4.9. *Suppose* $\rho_1 = 2$. *Then*

$$c_i = [i][d-i-1]\left(\frac{c_1}{[d-2]} - \sum_{k=2}^{i} \mu_k\right) \quad (1 \leqslant i \leqslant d-2). \tag{38}$$

Proof. We show (38) by induction. Clearly (38) holds for $i = 1$, so we assume $2 \leqslant i \leqslant d-2$. Applying (29) to v_i,

$$L^{(0)}Rv_i = \frac{[i][d-i-1]}{[i-1][d-i]} RL^{(0)}v_i - [i][d-i-1]\mu_i v_i.$$

This implies

$$c_i = \frac{[i][d-i-1]}{[i-1][d-i]} c_{i-1} - [i][d-i-1]\mu_i.$$

By induction,

$$c_{i-1} = [i-1][d-i]\left(\frac{c_1}{[d-2]} - \sum_{k=2}^{i-1} \mu_k\right).$$

Now (38) follows. $\square$

Lemma 4.10. *Suppose* $\rho_1 \geqslant 3$. *Then* $L^{(0)}v_2$ *and* v_1 *are linearly independent*.

Proof. By way of contradiction, we assume $L^{(0)}v_2$ lies in the span Y of $\{v_1\}$. We set $W = \sum_{i=0}^{d-2} R^i Y$ and $Z = U^{(0)} + W$. Clearly Z is invariant under R and F_i $(0 \leqslant i \leqslant d)$. If Z is invariant under L, then Z is invariant under L, R and F_i $(0 \leqslant i \leqslant d)$, so that $W = V$ by Lemma 2.4. This contradicts our assumption $\rho_1 \geqslant 3$. So, it is enough to show that Z is invariant under L.

Observe that $W \subseteq U^{(1)}$ and $L^{(+)}U^{(1)} = 0$ by $h = 1$, so that $LW \subseteq L^{(-)}W + L^{(0)}W \subseteq U^{(0)} + L^{(0)}W$. We have $L^{(0)}RY \subseteq Y$ from our assumption $L^{(0)}v_2 \in Y$, and this implies $L^{(0)}W \subseteq W$ by Lemma 3.19. Observe that $L^{(+)}U^{(0)} \subseteq W$ by Lemma 4.6, so that $LU^{(0)} \subseteq L^{(-)}U^{(0)} + L^{(0)}U^{(0)} + L^{(+)}U^{(0)} \subseteq U^{(0)} + W$, since $L^{(-)}U^{(0)} = 0$. Therefore $U^{(0)} + W$ is invariant under L. $\square$

When $\rho_1 \geqslant 3$, we set

$$w_1 = L^{(0)} v_2, \qquad w_i = R^{i-1} w_1 \quad (2 \leqslant i \leqslant d-1).$$

Lemma 4.11. *Suppose $\rho_1 \geqslant 3$. Then v_i and w_i are linearly independent for $1 \leqslant i \leqslant d-1$.*

Proof. Follows from Lemmas 4.10 and 3.1. □

When $\rho_1 \geqslant 3$, we set

$$L^{(-)} w_{i+1} = f_i u_i \quad (0 \leqslant i \leqslant d-2).$$

Lemma 4.12. *Suppose $\rho_1 \geqslant 3$. Then*

$$f_i = \frac{[d-i][d-i-1]}{[d][d-1]} f_0 \quad (0 \leqslant i \leqslant d-2). \tag{39}$$

Proof. Similar to the proof of Lemma 4.7. □

Lemma 4.13. *Suppose $\rho_1 \geqslant 3$. Then*

$$L^{(0)} v_{i+1} = m_i v_i + n_i w_i \quad (1 \leqslant i \leqslant d-2), \tag{40}$$

where

$$m_i = -[i][d-i-1] \sum_{k=2}^{i} \mu_k, \qquad n_i = \frac{[i][d-i-1]}{[d-2]}. \tag{41}$$

Proof. We show (40) by induction. Observe that (40) holds for $i = 1$ with $m_1 = 0$, $n_1 = 1$, since $L^{(0)} v_2 = w_1$. We assume $2 \leqslant i \leqslant d-2$. Applying (29) to v_i,

$$L^{(0)} R v_i = \frac{[i][d-i-1]}{[i-1][d-i]} R L^{(0)} v_i - [i][d-i-1] \mu_i v_i. \tag{42}$$

Observe that $L^{(0)} R v_i = L^{(0)} v_{i+1}$. By induction, we have

$$R L^{(0)} v_i = R(m_{i-1} v_{i-1} + n_{i-1} w_{i-1}) = m_{i-1} v_i + n_{i-1} w_i,$$

with

$$m_{i-1} = -[i-1][d-i] \sum_{k=2}^{i-1} \mu_k, \qquad n_{i-1} = \frac{[i-1][d-i]}{[d-2]}.$$

Thus (42) becomes

$$L^{(0)}v_{i+1} = \frac{[i][d-i-1]}{[i-1][d-i]}\left(-[i-1][d-i]\sum_{k=2}^{i-1}\mu_k v_i + \frac{[i-1][d-i]}{[d-2]}w_i\right)$$
$$- [i][d-i-1]\mu_i v_i$$
$$= -[i][d-i-1]\sum_{k=2}^{i}\mu_k v_i + \frac{[i][d-i-1]}{[d-2]}w_i. \qquad \square$$

Lemma 4.14. *Suppose* $\rho_1 = 3$. *Then* $\{v_i, w_i\}$ *is a basis of* $U_i^{(1)}$ $(1 \leqslant i \leqslant d-1)$.

Proof. Follows from Lemmas 4.11 and 3.2. $\square$

Hence, when $\rho_1 = 3$,

$$L^{(0)}w_{i+1} = s_i v_i + t_i w_i \quad (1 \leqslant i \leqslant d-2) \tag{43}$$

holds for some scalars s_i, t_i.

Lemma 4.15. *Suppose* $\rho_1 = 3$. *Then*

$$s_i = \frac{[i][d-i-1]}{[d-2]}s_1 \quad (1 \leqslant i \leqslant d-2), \tag{44}$$

$$t_i = [i][d-i-1]\left(\frac{t_1}{[d-2]} - \sum_{k=2}^{i}\mu_k\right) \quad (1 \leqslant i \leqslant d-2). \tag{45}$$

Proof. Similar to the proof of Lemma 4.13. $\square$

Lemma 4.16. *The following hold with the values of* a_i, b_i, e_i, f_i, m_i, n_i *given by* (32), (33), (36), (39) *and* (41).

(i) $Lu_0 = 0$, $Lu_1 = a_0u_0$, $Lu_{i+1} = a_iu_i + b_iv_i$ $(1 \leqslant i \leqslant d-1)$,
(ii) $Lv_1 = e_0u_0$, $Lv_{i+1} = e_iu_i + m_iv_i + n_iw_i$ $(1 \leqslant i \leqslant d-2)$,
(iii) $L^{(-)}w_{i+1} = f_iu_i$ $(0 \leqslant i \leqslant d-2)$.

Proof. Follows from (25) and Lemmas 4.3, 4.6, 4.7, 4.12 and 4.13. $\square$

5. Proof of $\rho_1 \leqslant 3$

In this section, we show $\rho_1 \leqslant 3$. By way of contradiction, we assume $\rho_1 \geqslant 4$.

Lemma 5.1. *The vectors* $v_1, w_1, L^{(0)}w_2$ *are linearly independent.*

Proof. Suppose $v_1, w_1, L^{(0)}w_2$ are linearly dependent. Since v_1, w_1 are linearly independent by Lemma 4.11, $L^{(0)}w_2$ lies in $Y = \text{span}\{v_1, w_1\}$. Observe that $RY = \text{span}\{v_2, w_2\}$ and $L^{(0)}v_2 = w_1 \in Y$, so that $L^{(0)}RY \subseteq Y$. Hence the subspace $W = \sum_{i=0}^{d-2} R^i Y$ is invariant under $L^{(0)}$ by Lemma 3.19. This implies $LW \subseteq L^{(-)}W + L^{(0)}W \subseteq U^{(0)} + W$. Moreover, $L^{(+)}U^{(0)} \subseteq W$ by Lemma 4.6. Hence $U^{(0)} + W$ is invariant under L. Clearly $U^{(0)} + W$ is invariant under R and F_i $(0 \leqslant i \leqslant d)$. These imply $U^{(0)} + W = V$ by Lemma 2.4, so that $U_1 = \{u_1, v_1, w_1\}$, contradicting our assumption $\rho_1 \geqslant 4$. $\square$

We set $L^{(0)}w_2 = x_1$, so that

$$Lw_2 = f_1u_1 + x_1. \tag{46}$$

Observe that u_1, v_1, w_1, x_1 are linearly independent by (19) and Lemma 5.1. Applying (14) to u_3,

$$L^3Ru_3 - [3]L^2RLu_3 + [3]LRL^2u_3 - RL^3u_3 - [3]\varepsilon_1L^2u_3 = 0. \tag{47}$$

We compute each term of (47) using Lemma 4.16 and (46). We need to divide our computation into two cases. First we consider the case of $d = 3$. Observe that $Ru_3 = Rv_2 = 0$ by Lemma 3.1.

$$\begin{aligned}
L^3Ru_3 &= 0,\\
L^2RLu_3 &= L^2R(a_2u_2 + b_2v_2) = L^2(a_2u_3) = L(a_2(a_2u_2 + b_2v_2))\\
&= L(a_2a_2u_2 + a_2b_2v_2) = a_2a_2(a_1u_1 + v_1) + a_2b_2(e_1u_1 + w_1)\\
&= (a_1a_2a_2 + e_1a_2b_2)u_1 + a_2a_2v_1 + a_2b_2w_1,\\
LRL^2u_3 &= LRL(a_2u_2 + b_2v_2) = LR(a_2(a_1u_1 + v_1) + b_2(e_1u_1 + w_1))\\
&= LR((a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1)\\
&= L((a_1a_2 + e_1b_2)u_2 + a_2v_2 + b_2w_2)\\
&= (a_1a_2 + e_1b_2)(a_1u_1 + v_1) + a_2(e_1u_1 + w_1) + b_2(f_1u_1 + x_1)\\
&= (a_1(a_1a_2 + e_1b_2) + e_1a_2 + f_1b_2)u_1 + (a_1a_2 + e_1b_2)v_1\\
&\quad + a_2w_1 + b_2x_1,\\
RL^3u_3 &\in \text{span}\{u_1\},\\
L^2u_3 &= (a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1.
\end{aligned}$$

Observe that the coefficient of x_1 in (47) becomes $[3]b_2$, so that $[3]b_2 = 0$, contradicting our assumption that q is not a root of unity. Next we consider the case of $d \geqslant 4$.

$$
\begin{aligned}
L^3Ru_3 &= L^3u_4 = L^2(a_3u_3 + b_3v_3) \\
&= L(a_3(a_2u_2 + b_2v_2) + b_3(e_2u_2 + m_2v_2 + n_2w_2)) \\
&= L((a_2a_3 + e_2b_3)u_2 + (b_2a_3 + m_2b_3)v_2 + n_2b_3w_2) \\
&= (a_2a_3 + e_2b_3)(a_1u_1 + v_1) \\
&\quad + (b_2a_3 + m_2b_3)(e_1u_1 + w_1) + n_2b_3(f_1u_1 + x_1), \\
L^2RLu_3 &= L^2R(a_2u_2 + b_2v_2) = L^2(a_2u_3 + b_2v_3) \\
&= L(a_2(a_2u_2 + b_2v_2) + b_2(e_2u_2 + m_2v_2 + n_2w_2)) \\
&= L((a_2a_2 + b_2e_2)u_2 + (a_2b_2 + b_2m_2)v_2 + b_2n_2w_2) \\
&= (a_2a_2 + b_2e_2)(a_1u_1 + v_1) + (a_2b_2 + b_2m_2)(e_1u_1 + w_1) \\
&\quad + b_2n_2(f_1u_1 + x_1), \\
LRL^2u_3 &= LRL(a_2u_2 + b_2v_2) = LR(a_2(a_1u_1 + v_1) + b_2(e_1u_1 + w_1)) \\
&= LR((a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1) \\
&= L((a_1a_2 + e_1b_2)u_2 + a_2v_2 + b_2w_2) \\
&= (a_1a_2 + e_1b_2)(a_1u_1 + v_1) + a_2(e_1u_1 + w_1) + b_2(f_1u_1 + x_1), \\
RL^3u_3 &\in \operatorname{span}\{u_1\}, \\
L^2u_3 &= (a_1a_2 + e_1b_2)u_1 + a_2v_1 + b_2w_1.
\end{aligned}
$$

Now looking at the coefficients of x_1 in (47),

$$n_2b_3 - [3]b_2n_2 + [3]b_2 = 0, \tag{48}$$

so that

$$\frac{[2][d-3]}{[d-2]} \cdot \frac{[4][3]}{[2]} - [3][3] \cdot \frac{[2][d-3]}{[d-2]} + [3][3] = \frac{[3][d]}{[d-2]} = 0,$$

contradicting our assumption that q is not a root of unity. This completes the proof of $\rho_1 = 3$.

6. Proof of $d = 3$

In this section, we assume $\rho_1 = 3$, and we show $d = 3$. By way of contradiction, we assume $d \geqslant 4$. Applying (14) to v_3,

$$L^3Rv_3 - [3]L^2RLv_3 + [3]LRL^2v_3 - RL^3v_3 - [3]\varepsilon_1v_3 = 0. \tag{49}$$

We compute each term of (49) using Lemmas 4.16 and 3.17. The term of L^3Rv_3 vanishes when $d = 4$. When $d \geqslant 5$, it becomes

$$
\begin{aligned}
L^3Rv_3 &= L^3v_4 = L^2(e_3u_3 + m_3v_3 + n_3w_3)\\
&= L(e_3(a_2u_2 + b_2v_2) + m_3(e_2u_2 + m_2v_2 + n_2w_2)\\
&\quad + n_3(f_2u_2 + s_2v_2 + t_2w_2))\\
&= L((a_2e_3 + e_2m_3 + f_2n_3)u_2 + (b_2e_3 + m_2m_3 + s_2n_3)v_2\\
&\quad + (n_2m_3 + t_2n_3)w_2)\\
&= (a_2e_3 + e_2m_3 + f_2n_3)(a_1u_1 + v_1)\\
&\quad + (b_2e_3 + m_2m_3 + s_2n_3)(e_1u_1 + w_1)\\
&\quad + (n_2m_3 + t_2n_3)(f_1u_1 + s_1v_1 + t_1w_1)\\
&= (a_1(a_2e_3 + e_2m_3 + f_2n_3) + e_1(b_2e_3 + m_2m_3 + s_2n_3)\\
&\quad + f_1(n_2m_3 + t_2n_3))u_1\\
&\quad + ((a_2e_3 + e_2m_3 + f_2n_3) + s_1(n_2m_3 + t_2n_3))v_1\\
&\quad + ((b_2e_3 + m_2m_3 + s_2n_3) + t_1(n_2m_3 + t_2n_3))w_1.
\end{aligned}
$$

The other terms become

$$
\begin{aligned}
L^2RLv_3 &= L^2R(e_2u_2 + m_2v_2 + n_2w_2) = L^2(e_2u_3 + m_2v_3 + n_2w_3)\\
&= L(e_2(a_2u_2 + b_2v_2) + m_2(e_2u_2 + m_2v_2 + n_2w_2)\\
&\quad + n_2(f_2u_2 + s_2v_2 + t_2w_2))\\
&= L((a_2e_2 + e_2m_2 + f_2n_2)u_2 + (b_2e_2 + m_2m_2 + s_2n_2)v_2\\
&\quad + (m_2n_2 + t_2n_2)w_2)\\
&= (a_2e_2 + e_2m_2 + f_2n_2)(a_1u_1 + v_1)\\
&\quad + (b_2e_2 + m_2m_2 + s_2n_2)(e_1u_1 + w_1)\\
&\quad + (m_2n_2 + t_2n_2)(f_1u_1 + s_1v_1 + t_1w_1)\\
&= (a_1(a_2e_2 + e_2m_2 + f_2n_2) + e_1(b_2e_2 + m_2m_2 + s_2n_2)\\
&\quad + f_1(m_2n_2 + t_2n_2))u_1\\
&\quad + ((a_2e_2 + e_2m_2 + f_2n_2) + s_1(m_2n_2 + t_2n_2))v_1\\
&\quad + ((b_2e_2 + m_2m_2 + s_2n_2) + t_1(m_2n_2 + t_2n_2))w_1,
\end{aligned}
$$

$$
\begin{aligned}
LRL^2v_3 &= LRL(e_2u_2 + m_2v_2 + n_2w_2)\\
&= LR(e_2(a_1u_1 + v_1) + m_2(e_1u_1 + w_1) + n_2(f_1u_1 + s_1v_1 + t_1w_1))\\
&= LR((a_1e_2 + e_1m_2 + f_1n_2)u_1 + (e_2 + s_1n_2)v_1 + (m_2 + t_1n_2)w_1)\\
&= L((a_1e_2 + e_1m_2 + f_1n_2)u_2 + (e_2 + s_1n_2)v_2 + (m_2 + t_1n_2)w_2)\\
&= (a_1e_2 + e_1m_2 + f_1n_2)(a_1u_1 + v_1) + (e_2 + s_1n_2)(e_1u_1 + w_1)\\
&\quad + (m_2 + t_1n_2)(f_1u_1 + s_1v_1 + t_1w_1)\\
&= (a_1(a_1e_2 + e_1m_2 + f_1n_2) + e_1(e_2 + s_1n_2) + f_1(m_2 + t_1n_2))u_1\\
&\quad + ((a_1e_2 + e_1m_2 + f_1n_2) + s_1(m_2 + t_1n_2))v_1\\
&\quad + ((e_2 + s_1n_2) + t_1(m_2 + t_1n_2))w_1,\\
RL^3v_3 &\in \operatorname{span}\{u_1\},\\
L^2v_3 &= (a_1e_2 + e_1m_2 + f_1n_2)u_1 + (e_2 + s_1n_2)v_1 + (m_2 + t_1n_2)w_1.
\end{aligned}
$$

When $d \geqslant 5$, by a routine computation, the coefficient of w_1 in (49) becomes

$$\begin{aligned}&b_2e_3 + m_2m_3 + s_2n_3 + t_1(n_2m_3 + t_2n_3) - [3]((b_2e_2 + m_2m_2 + s_2n_2)\\&\quad + t_1(m_2n_2 + t_2n_2)) + [3](e_2 + s_1n_2 + t_1(m_2 + t_1n_2))\\&\quad - [3]\varepsilon_1(m_2 + t_1n_2) = -\frac{[3][d-3]}{[d-1]}e_0,\end{aligned}$$

so that $e_0 = 0$, and this implies $e_i = 0$ $(1 \leqslant i \leqslant d-1)$. The coefficient of v_1 becomes

$$\begin{aligned}&f_2n_3 + s_1(n_2m_3 + t_2n_3) - [3](f_2n_2 + s_1(m_2n_2 + t_2n_2))\\&\quad + [3](f_1n_2 + s_1(m_2 + t_1n_2)) - [3]\varepsilon_1s_1n_2 = \frac{[3][d-3]}{[d-1]}f_0,\end{aligned}$$

so that $f_0 = 0$. When $d = 4$, the coefficient of w_1 becomes

$$\begin{aligned}&-[3]((b_2e_2 + m_2m_2 + s_2n_2) + t_1(m_2n_2 + t_2n_2))\\&\quad + [3](e_2 + s_1n_2 + t_1(m_2 + t_2n_2)) - [3]\varepsilon_1(m_2 + t_1n_2) = -e_0,\end{aligned}$$

so that $e_i = 0$ $(1 \leqslant i \leqslant d-1)$. The coefficient of v_1 becomes

$$\begin{aligned}&-[3](f_2n_2 + s_1(m_2n_2 + t_2n_2)) + [3](f_1n_2 + s_1(m_2 + t_1n_2))\\&\quad - [3]\varepsilon_1s_1n_2 = f_0.\end{aligned}$$

In either case, $e_i = f_i = 0$ $(1 \leqslant i \leqslant d-2)$, so that $L^{(-)}U^{(1)} = 0$ and hence $LU^{(1)} \subseteq U^{(1)}$. Since $U^{(1)}$ is invariant under R and F_i $(0 \leqslant i \leqslant d)$, we get $U^{(1)} = V$ by Lemma 2.4, a contradiction. This completes the proof of Theorem 1.3.

7. Proof of Theorems 1.5–1.9

Proof of Theorem 1.5. Follows from Lemmas 2.1, 3.3, 4.2 and 4.5. □

Lemma 7.1. *Suppose Theorem 1.3*(i) *holds. Then the maps R, L act on the basis* (3) *as follows.*

$$\begin{aligned}&Ru_i = u_{i+1} \quad (0 \leqslant i \leqslant d-1),\\&Ru_d = 0,\\&Rv_i = v_{i+1} \quad (1 \leqslant i \leqslant d-2),\\&Lu_0 = 0,\\&Lu_1 = a_0u_0,\\&Lu_{i+1} = a_iu_i + b_iv_i \quad (1 \leqslant i \leqslant d-1),\end{aligned}$$

$$Lv_1 = e_0u_0,$$
$$Lv_{i+1} = e_iu_i + c_iv_i \quad (1 \leqslant i \leqslant d-2),$$

where the coefficients satisfy (32), (33), (36) *and* (38).

Proof. Follows from Theorem 1.5 and Eqs. (24), (31)–(38). □

Proof of Theorem 1.6. First observe the following formulas hold, which can be verified by routine computations.

$$\sum_{k=1}^{i} \mu_k = [i]\eta_{d+i+1}, \tag{50}$$

$$\sum_{k=2}^{i} \mu_k = [i-1]\eta_{d+i+2}. \tag{51}$$

Now the expressions for a_i, b_i, c_i follow from Lemma 7.1. Applying (14) to v_2,

$$L^3Rv_2 - [3]L^2RLv_2 + [3]LRL^2v_2 - RL^3v_2 - [3]\varepsilon_0v_2 = 0. \tag{52}$$

We compute each term of (52) as follows using Lemma 7.1.

$$\begin{aligned}
L^3Rv_2 &= L^3v_3 = L^2(e_2u_2 + c_2v_2)\\
&= L(e_2(a_1u_1 + v_1) + c_2(e_1u_1 + c_1v_1))\\
&= L((a_1e_2 + e_1c_2)u_1 + (e_2 + c_1c_2)v_1)\\
&= (a_0(a_1e_2 + e_1c_2) + e_0(e_2 + c_1c_2))u_0,\\
L^2RLv_2 &= L^2R(e_1u_1 + c_1v_1) = L^2(e_1u_2 + c_1v_2)\\
&= L(e_1(a_1u_1 + v_1) + c_1(e_1u_1 + c_1v_1))\\
&= L((a_1e_1 + c_1e_1)u_1 + (e_1 + c_1c_1)v_1)\\
&= (a_0(a_1e_1 + c_1e_1) + e_0(e_1 + c_1c_1))u_0,\\
LRL^2v_2 &= LRL(e_1u_1 + c_1v_1) = LR(a_0e_1 + e_0c_1)u_0\\
&= a_0(a_0e_1 + e_0c_1)u_0,\\
RL^3 &= 0,\\
L^2v_2 &= (a_0e_1 + e_0c_1)u_0.
\end{aligned}$$

Hence (52) implies

$$\begin{aligned}
&a_0(a_1e_2 + e_1c_2) + e_0(e_2 + c_1c_2) - [3](a_0(a_1e_1 + c_1e_1) + e_0(e_1 + c_1c_1))\\
&\quad + [3]a_0(a_0e_1 + e_0c_1) - [3]\varepsilon_0(a_0e_1 + e_0c_1) = 0.
\end{aligned}$$

Using Lemma 7.1, we get an equation in terms of a_0, c_1 and e_0, in which e_0 has degree one, so that we may solve it in e_0. After a routine computation, we get

$$\begin{aligned} e_0 = &-\frac{[d-1]}{[2][d]}a_0^2 - \frac{[d][d-1]}{[2][d-2]^2}c_1^2 + \frac{[4][d-1]}{[2]^2[d-2]}a_0c_1 \\ &+ [d-1]\,\eta_{d+3}a_0 - \frac{[d][d-1]}{[d-2]}\eta_{d+1}c_1. \end{aligned}$$

This implies the expression for e_i in Theorem 1.6. □

Proof of Theorem 1.8. Follows from Theorem 1.6 and the definition of R, L. □

Proof of Theorem 1.9. Let R, L denote the maps which act on the given basis as in Theorem 1.6. Define subspaces $U_0, \ldots\ U_d$ by

$$\begin{aligned} U_0 &= \operatorname{span}\{u_0\}, \\ U_i &= \operatorname{span}\{u_i, v_i\} \quad (1 \leqslant i \leqslant d-1), \\ U_d &= \operatorname{span}\{u_d\}. \end{aligned}$$

Observe that V is decomposed into direct sum of $U_0, \ldots, U_d$. Let $F_i : V \longrightarrow U_i$ denote the projection. By Lemmas 2.7 and 2.8, it is enough to show that R, L satisfy the relations (13) and (14).

Let C_1 denote the left side of (13). It is routine to verify that $C_1u_i = 0$ $(0 \leqslant i \leqslant d-2)$ and $C_1v_i = 0$ $(1 \leqslant i \leqslant d-2)$. So (13) holds.

Let C_2 denote the left side of (14). We shold verify $C_2u_i = 0$ for $2 \leqslant i \leqslant d$ and $C_2v_i = 0$ for $2 \leqslant i \leqslant d-1$. When $4 \leqslant i \leqslant d-2$, after routine computation,

$$\begin{aligned} C_2u_i = (&(a_ia_{i-1} + b_ie_{i-1})a_{i-2} + (a_ib_{i-1} + b_ic_{i-1})e_{i-2} \\ &- [3]((a_{i-1}a_{i-1} + b_{i-1}e_{i-1})a_{i-2} + (a_{i-1}b_{i-1} + b_{i-1}c_{i-1})e_{i-2}) \\ &+ [3]((a_{i-1}a_{i-2} + b_{i-1}e_{i-2})a_{i-2} + (a_{i-1}b_{i-2} + b_{i-1}c_{i-2})e_{i-2}) \\ &- ((a_{i-1}a_{i-2} + b_{i-1}e_{i-2})a_{i-3} + (a_{i-1}b_{i-2} + b_{i-1}c_{i-2})e_{i-3}) \\ &- [3]\varepsilon_{i-2}(a_{i-1}a_{i-2} + b_{i-1}e_{i-2}))u_{i-2} \\ &+ ((a_ia_{i-1} + b_ie_{i-1})b_{i-2} + (a_ib_{i-1} + b_ic_{i-1})c_{i-2} \\ &- [3]((a_{i-1}a_{i-1} + b_{i-1}e_{i-1})b_{i-2} + (a_{i-1}b_{i-1} + b_{i-1}c_{i-1})c_{i-2}) \\ &+ [3]((a_{i-1}a_{i-2} + b_{i-1}e_{i-2})b_{i-2} + (a_{i-1}b_{i-2} + b_{i-1}c_{i-2})c_{i-2}) \\ &- ((a_{i-1}a_{i-2} + b_{i-1}e_{i-2})b_{i-3} + (a_{i-1}b_{i-2} + b_{i-1}c_{i-2})c_{i-3}) \\ &- [3]\varepsilon_{i-2}(a_{i-1}b_{i-2} + b_{i-1}c_{i-2}))v_{i-2}. \end{aligned}$$

It is routine to verify that both coefficients vanish, so that $C_2u_i = 0$. In the same way, we can verify $C_2v_i = 0$. Similarly, we can verify $C_2u_i = 0$ and $C_2v_i = 0$ for the case of $i = 2, 3, d-1, d$. □

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